Int. J. Solids Structures Vol. 18, No. 12, pp. 1075–1082, 1982
Printed in Great Britain.

0020-7683/82/121075–08$03.00/0

A THREE-DIMENSIONAL RECTANGULAR CRACK SUBJECTED TO SHEAR LOADING

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(*Received* 5 *March* 1982)

Abstract—In this paper a solution is derived to treat the three-dimensional elastostatic problem of a narrow rectangular crack embedded in an infinite elastic medium and subjected to equal and opposite shear stress distribution across its faces. Employing two-dimensional integral transforms and assuming a plane-strain solution across the width of the crack, the stress field ahead of the crack length is reduced to the solution of an integral equation of Fredholm type. A numerical solution of the integral equation and the corresponding mode II stress-intensity factor is obtained for several crack dimensions and Poisson's ratios of the material.

INTRODUCTION

Reference[1] contains an integral transform solution which reduce the problem of determining the mode I stress-intensity factor of a narrow rectangular crack embedded in an infinite solid to the solution of a standard integral equation. The solution consists of representing the components of stress and displacement in terms of double integrals containing an auxiliary function. By assuming a plane-strain stress field across the width of the crack, the auxiliary function, which determine the stress field ahead of the crack length, is shown to be governed by a Fredholm integral equation of the second kind.

This paper treats the accompanying problem of finding the mode II stress-intensity factor for the same three-dimensional crack profile. It is assumed that the crack surfaces are deformed by the application of equal and opposite shear stresses parallel to one side of the crack. The solution employed leads to an integral equation of Fredholm type which is solved numerically to determine the dependence of the k_2-factor on Poisson's ratio of the material and on crack aspect ratios. In all cases examined the maximum value of k_2 is less than that of the corresponding plane-strain problem.

BASIC EQUATIONS AND FORMULATION

A crack of rectangular planform (sides $2a \times 2b$) is embedded in the mid-plane of a three-dimensional, homogeneous, elastic and isotropic solid. In terms of cartesian coordinates (x, y, z) centered at the mid-point of the crack, the surfaces of the crack are described by the relations $|x| \leq a$, $|y| \leq b$ in the $z = \mp 0$ planes. When the crack surfaces are deformed by the application of equal and opposite shears τ_{xz}, the stress-field can be found by considering the half-space $z \geq 0$ subject to the following boundary conditions on $z = 0$

$$\sigma_z = 0, \quad \text{all } x \text{ and } y, \tag{1}$$

$$\tau_{xz} = \tau_0(x, y), \quad |x| \leq a, \quad |y| \leq b \tag{2a}$$

$$\tau_{yz} = 0, \quad |x| \leq a, \quad |y| \leq b \tag{2b}$$

$$u_x = 0, \quad |x| > a, \quad |y| > b \tag{2c}$$

and the usual regularity requirements at locations away from the crack region. In eqns (1) and (2), (u_x, u_y, u_z) designate the components of the displacement vector, σ_z, τ_{xz} and τ_{yz} denote the components of the traction on the $z = 0$ plane and $\tau_0(x, y)$ stands for the prescribed shear stress inside the crack region.

A suitable potential function representation of the displacement components which satisfy the field equations of equilibrium and automatically confirm the condition in equation (1) can be expressed as [2, 3]

$$2\mu u_x = -2(1-\nu)\frac{\partial g}{\partial z} + z\frac{\partial^2 g}{\partial x^2}, \tag{3a}$$

$$2\mu u_y = z\frac{\partial^2 g}{\partial y \partial z}, \tag{3b}$$

$$2\mu u_z = -(1-2\nu)\frac{\partial g}{\partial x} + z\frac{\partial^2 g}{\partial x \partial z}, \tag{3c}$$

where μ and ν designate the shearing modulus and Poisson's ratio of the material, respectively, and $g(x, y, z)$ is a harmonic function satisfying Laplace's equation in three dimensions

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0. \tag{4}$$

Some of the associated stresses are

$$\sigma_z = z\frac{\partial^2 g}{\partial z^2}, \tag{5a}$$

$$\tau_{zx} = -(1-\nu)\frac{\partial^2 g}{\partial z^2} + \nu\frac{\partial^2 g}{\partial x^2} + z\frac{\partial^3 g}{\partial x^2 \partial y}, \tag{5b}$$

$$\tau_{zy} = \nu\frac{\partial^2 g}{\partial x \partial y} + z\frac{\partial^3 g}{\partial x \partial y \partial z}. \tag{5c}$$

For the class of problems under discussion, the substitution $g = XYZ$ leads to the variable separable solution

$$g(x, y, z) = \int_0^\infty \int_0^\infty \frac{B(\xi, \eta)}{\sqrt{\xi^2+\eta^2}} \cos(x\xi)\cos(y\eta)\, e^{-(\xi^2+\eta^2)^{1/2} z}\, d\xi\, d\eta, \tag{6}$$

in which $B(\xi, \eta)$ is an unknown function and the factor $1/(\xi^2+\eta^2)^{1/2}$ has been introduced for convenience. Inserting eqn (6) into eqns (3)–(5), the following expressions are reached:

$$2\mu u_x(x, y, 0) = 2(1-\nu)\int_0^\infty \int_0^\infty \frac{B(\xi, \eta)}{\sqrt{\xi^2+\eta^2}} \cos(x\xi)\cos(y\eta)\, d\xi\, d\eta, \tag{7a}$$

$$\tau_{zx}(x, y, 0) = -\int_0^\infty \int_0^\infty \frac{\xi^2 + (1-\nu)\eta^2}{\xi^2+\eta^2} B(\xi, \eta)\cos(x\xi)\cos(y\eta)\, d\xi\, d\eta, \tag{7b}$$

$$\tau_{zy}(x, y, 0) = \nu\int_0^\infty \int_0^\infty \frac{\xi\eta}{\xi^2+\eta^2} B(\xi, \eta)\sin(x\xi)\sin(y\eta)\, d\xi\, d\eta. \tag{7c}$$

Making use of these expressions in conjunction with boundary conditions (2) yield the necessary relations to determine the unknown function $B(\xi, \eta)$. Inside the crack region ($|x| \le a$, $|y| \le b$)

$$\int_0^\infty \int_0^\infty \left(1 - \frac{\nu\eta^2}{\xi^2+\eta^2}\right) B(\xi, \eta)\cos(x\xi)\cos(y\eta)\, d\xi\, d\eta = -\tau_0(x, y), \tag{8a}$$

$$\int_0^\infty \int_0^\infty \frac{\xi\eta}{\xi^2+\eta^2} B(\xi, \eta)\sin(x\xi)\sin(y\eta)\, d\xi\, d\eta = 0, \tag{8b}$$

while outside the crack the following relation holds

$$\int_0^\infty \int_0^\infty \frac{B(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} \cos(x\xi)\cos(y\eta)\,d\xi\,d\eta = 0, \quad |x|>a, \quad |y|>b. \tag{8c}$$

REDUCTION TO AN INTEGRAL EQUATION

In order to derive the integral equation governing the unknown function $B(\xi, \eta)$, it is expedient to express the shear stress $\tau_{xz}(x, y, 0)$ in the form

$$\tau_{xz}(x, y, 0) = \tau(x)s(y), \tag{9}$$

in which $\tau(x)$ and $s(y)$ are arbitrary functions yet to be determined. Inside the crack these function are specified while in the outside region they yield the mode II stress-intensity factor. Applying relation (9) to the expression in eqn (7b) and assuming the inversion of the two-dimensional Fourier cosine transform, it is found that

$$B(\xi,\eta) = -\frac{4}{\pi^2}\frac{\xi^2+\eta^2}{\xi^2+(1-\nu)\eta^2}\tau_c(\xi)s_c(\eta), \tag{10}$$

where $\tau_c(\xi)$ and $s_c(\eta)$ designate the one-dimensional cosine transforms of the functions $\tau(x)$ and $s(y)$ respectively, i.e.

$$\tau_c(\xi) = \int_0^\infty \tau(x)\cos(x\xi)\,dx, \tag{11a}$$

$$s_c(\eta) = \int_0^\infty s(y)\cos(y\eta)\,dy. \tag{11b}$$

Since the functions $\tau(x)$ and $s(y)$ are specified inside the crack

$$\tau(x) = \tau_0(x) \qquad 0 \leq x \leq a, \tag{12a}$$

$$s(y) = s_0(y) \qquad 0 \leq y \leq b, \tag{12b}$$

it immediately follows that

$$\int_0^\infty \tau_c(\xi)\cos(x\xi)\,d\xi = \frac{\pi}{2}\tau_0(x), \quad 0 \leq x \leq a, \tag{13a}$$

$$\int_0^\infty s_c(\eta)\cos(y\eta)\,d\eta = \frac{\pi}{2}s_0(y), \quad 0 \leq y \leq b. \tag{13b}$$

For the purpose of determining the corresponding relations governing $\tau_c(\xi)$ and $s_c(\eta)$ in the region of the $z = 0$ plane outside the crack, relation (10) is inserted into equation (7a) and upon setting $y = 0$ (in order to compute the maximum stress-intensity factor along the side of the crack) it is found that

$$u_x = -\frac{2(1-\nu)}{\pi\mu}\int_0^\infty \tau_c(\xi)s_k(\xi)\cos(x\xi)\,d\xi, \quad x > a. \tag{14}$$

In eqn (14), the following abbreviation has been introduced

$$s_k(\xi) = \frac{2}{\pi}\int_0^\infty \frac{(\xi^2+\eta^2)^{1/2}}{\xi^2+(1-\nu)\eta^2}s_c(\eta)\,d\eta, \tag{15}$$

which may be simplified by substituting eqn (11b) and making a permissible interchange in the order of integration to read

$$s_k(\xi) = \frac{2}{\pi}\int_0^\infty s(\theta)I(\xi, \theta)\,d\theta, \tag{16}$$

with

$$I(s, \theta) = \frac{1}{1-\nu}\int_0^\infty \frac{\cos(\eta\theta)\,d\eta}{(\xi^2+\eta^2)^{1/2}} \sum_{n=0}^\infty \left(\frac{\nu}{\nu-1}\right)^n \frac{\xi^{2n}}{(\xi^2+\eta^2)^{2n}}. \tag{17}$$

Utilizing the identity[4]

$$\int_0^\infty \frac{\cos(\eta\theta)\,d\eta}{(\xi^2+\eta^2)^{n+1/2}} = \frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\left(\frac{\theta}{2\xi}\right)^n K_n(\theta\xi). \tag{18}$$

The integral in eqn (17) may now be written in the form

$$I(s, \theta) = \frac{\sqrt{\pi}}{1-\nu}\sum_{n=0}^\infty \left[\frac{\nu\xi\theta}{2(\nu-1)}\right]^n \frac{K_n(\theta\xi)}{\Gamma(n+\frac{1}{2})}, \tag{19}$$

where K_n is the modified Bessel function of the second kind of order n. In this manner, condition (2c) when used in the relation (14) yields

$$\int_0^\infty \tau_c(\xi)s_k(\xi)\cos(x\xi)\,d\xi = 0, \quad x > a, \tag{20}$$

which is the required equation satisfied by $\tau_c(\xi)$ in the region outside the crack. In a similar manner the equation governing the function $s_c(\eta)$ along the y-axis is

$$\int_0^\infty s_c(\eta)\tau_k(\eta)\cos(y\eta)\,d\eta = 0, \quad y > b. \tag{21}$$

Here, the function $\tau_k(\eta)$ stands for

$$\tau_k(\eta) = \int_0^\infty \tau(\theta)I^*(\theta, \eta)\,d\theta, \tag{22}$$

and

$$I^*(\theta, \eta) = \frac{\sqrt{\pi}}{1-\nu}\sum_{n=0}^\infty \left[\frac{\nu\theta\eta}{2(\nu-1)}\right]^n \frac{K_n(\theta\eta)}{\Gamma(n+\frac{1}{2})}. \tag{23}$$

Equations (13a) and (20) form a standard set of dual integral equations with an arbitrary weight function. It is readily solved by writing

$$\tau_c(\xi)s_k(\xi) = \int_0^a \phi(t)J_0(\xi t)\,dt, \tag{24}$$

where J_0 is the Bessel function of the first kind of order zero. The auxiliary function, $\phi(t)$, is determined by way of the Fredholm equation

$$\phi(t) + \int_0^a \phi(\theta)K(\theta, t)\,d\theta = -t\int_0^t \frac{\tau_0(x)\,dx}{(t^2-x^2)^{1/2}}, \tag{25}$$

and the kernel, $K(\theta, t)$, is given by

$$K(\theta, t) = t \int_0^\infty y \left[\frac{1}{y s_k(y)} - 1 \right] J_0(yt) J_0(y\theta)\, dy. \tag{26}$$

With view toward numerical treatment, it is found convenient to rewrite eqns (25) and (26) by introducing the non-dimensional parameters

$$t = ar, \quad \theta = a\rho, \quad z = ay, \tag{27}$$

and, furthermore, when the crack is deformed by means of the application of a constant shearing stress, $\tau_0(x) = \tau_0$, the following contraction is introduced

$$\phi(t) = \phi(ar) = -\tau_0 a \sqrt{r}\, \Phi(r) \cdot \frac{\pi}{2}, \tag{28}$$

so as to enable one to write the integral equation in the standard form

$$\Phi(r) + \int_0^1 \Phi(\rho) L(ar, a\rho)\, d\rho = r^{1/2} \tag{29}$$

in which the symmetrical kernel is given by

$$L(ar, a\rho) = (r\rho)^{1/2} \int_0^\infty z \left[\frac{1}{\left(\frac{z}{a}\right) s_k\left(\frac{z}{a}\right)} - 1 \right] J_0(z\rho) J_0(zr)\, dz, \tag{30}$$

and $s_k(z/a)$ is given in eqns (16)–(19). The next step in the analysis is to determine the function $s(\theta)$ in eqn (16). This will be done next by assuming a plane-strain solution across the width of the crack.

PLANE STRAIN SOLUTION

For a narrow rectangular crack a plane strain solution can be assumed across the width of the crack to furnish the stress field ahead of the crack length. Utilizing this assumption the function $s(y)$ is computed by evaluating $\tau_{zy}(y, z)$ at $z = 0$. In the yz-plane the appropriate crack conditions are

$$\sigma_z(y, 0) = 0, \quad \text{all values of } y, \tag{31a}$$

$$\tau_{zy}(y, 0) = s_0(y), \quad 0 \leq y \leq b, \tag{31b}$$

$$u_y(y, 0) = 0, \quad y > b \tag{31c}$$

and the solution of this class of problems can be achieved by expressing the displacements u_y and u_z in terms of a potential function $h(y, z)$ as

$$2\mu u_y = -2(1-\nu)\frac{\partial h}{\partial z} + z\frac{\partial^2 h}{\partial y^2}, \tag{32a}$$

$$2\mu u_z = -(1-2\nu)\frac{\partial h}{\partial y} + z\frac{\partial^2 h}{\partial y \partial z}. \tag{32b}$$

Some of the associated stresses are readily computed as

$$\sigma_z(y, z) = z\frac{\partial^3 h}{\partial y \partial z^2}, \tag{33a}$$

$$\tau_{zy}(y, z) = -\frac{\partial^2 h}{\partial z^2} + z\frac{\partial^3 h}{\partial y^2 \partial z}. \tag{33b}$$

If the function $h(y, z)$ is assumed to be given by the expression

$$h = -\int_0^\infty t^{-1}\psi(t)\cos(ty)\,e^{-tz}\,dt, \tag{34}$$

then it is readily shown that boundary conditions (31b) and (31c) lead to the dual integral equations

$$\int_0^\infty t\psi(t)\cos(ty)\,dt = s_0(y), \quad 0 \leq y \leq b, \tag{35a}$$

$$\int_0^\infty \psi(t)\cos(ty)\,dt = 0, \quad y > b, \tag{35b}$$

where

$$\psi(t) = \frac{2}{\pi}\int_0^b \theta J_0(\theta t)\,d\theta \int_0^\theta \frac{s_0(y)\,dy}{(\theta^2 - y^2)^{1/2}}. \tag{36}$$

In order to evaluate $s_k(\xi)$ it immediately follows from eqns (16) and (19) that

$$s_k(\xi) = \frac{2^{1-n}}{(1-\nu)\sqrt{\pi}}\sum_{n=0}^\infty \frac{\left(\frac{\nu\xi}{\nu-1}\right)^n}{\Gamma(n+\frac{1}{2})}\int_0^\infty s(y)y^n K_n(y)\,dy. \tag{37}$$

Since

$$s(y) = \int_0^\infty t\psi(t)\cos(ty)\,dt, \tag{38}$$

and by inserting eqn (38) into (37), making a permissible change in the order of integration and in view of the result[4]

$$\int_0^\infty y^n K_n(y\xi)\cos(yt)\,dy = \frac{\sqrt{\pi}}{2^{1-n}}\frac{\xi^n\Gamma(n+\frac{1}{2})}{(t^2+\xi^2)^{n+1/2}}, \tag{39}$$

it can be shown that

$$s_k(\xi) = \frac{1}{1-\nu}\sum_{n=0}^\infty \left(\frac{\nu\xi^2}{\nu-1}\right)^n \int_0^\infty \frac{t\psi(t)\,dt}{(t^2+\xi^2)^{n+1/2}}. \tag{40}$$

Substituting the relation (36) into (40) and evaluating the inner integral with the help of the identity

$$\int_0^\infty \frac{tJ_0(st)\,dt}{(t^2+\xi^2)^{n+1/2}} = \frac{2^{1/2-n}}{\Gamma(\frac{1}{2}+n)}\left(\frac{s}{\xi}\right)^{n-1/2} K_{1/2-n}(s\xi), \tag{41}$$

it is readily shown that

$$s_k(\xi) = \frac{1}{\pi(1-\nu)}\sum_{n=0}^\infty \frac{2^{(3/2)-n}}{\Gamma(n+\frac{1}{2})}\left(\frac{\nu}{\nu-1}\right)^n \xi^{n+1/2}\int_0^b t^{1/2-n}K_{1/2-n}(t\xi)\,dt,$$
$$\times\int_0^t \frac{s_0(y)\,dy}{(t^2-y^2)^{1/2}}, \tag{42}$$

which determines the function s_k. For constant shear stress, $s_0(y) = 1$, and eqn (42) reduces to

$$\xi s_k(\xi) = 1 - \frac{1}{1-\nu}\sum_{n=0}^{\infty}\left(\frac{\nu}{\nu-1}\right)^n \frac{(b\xi)^n K_{n+1/2}(b\xi)}{2^{n-1/2}\Gamma(n+\frac{1}{2})}. \tag{43}$$

Inserting eqn (43) into (30) and after some simplification the kernel of the integral equation can be written as

$$L(ar, a\rho) = (r\rho)^{1/2}\int_0^{\infty}\left[\frac{\theta D\left(\frac{b}{a}\theta\right)}{1-D\left(\frac{b}{a}\theta\right)}\right]J_0(\theta r)J_0(\theta\rho)\,d\theta, \tag{44}$$

in which the following contraction has been made

$$D\left(\frac{b}{a}\theta\right) = \frac{1}{1-\nu}\sum_{n=0}^{\infty}\left(\frac{\nu}{\nu-1}\right)^n \cdot \frac{2^{1/2-n}\left(\frac{b}{a}\theta\right)^{n+1/2}}{\Gamma(\frac{1}{2}+n)} K_{1/2+n}\left(\frac{b}{a}\theta\right). \tag{45}$$

It is not difficult to confirm that the integral (44) is rapidly convergent throughout its range and eqn (29) can be solved numerically in a routine mannar[5]. In contrast to the opening mode loading[1], the kernel (and consequently the stress-intensity factor) in the present case is dependent on the material of the solid.

STRESS-INTENSITY FACTOR

In order to compute the stress-intensity factor, the relations (27) and (28) are inserted into (24) and the result may be expressed as

$$\tau_c(\xi)s_k(\xi) = -\frac{a\pi\tau_0}{2\xi}\left\{\Phi(1)J_1(a\xi) + \int_0^1 sJ_1(s\xi)\cdot \right.$$
$$\left. \times \frac{d}{ds}[s^{-1/2}\Phi(s)]\,ds\right\}, \tag{46}$$

Since from eqn (11a)

$$\tau(x) = \frac{2}{\pi}\int_0^{\infty}\tau_c(\xi)\cos(x\xi)\,d\xi, \tag{47}$$

it immediately follows that

$$\tau(x) = -\tau_0 a\Phi(1)\int_0^{\infty}\frac{J_1(a\xi)\cos(x\xi)}{\xi s_k(\xi)}\,d\xi + \cdots\cdots, \tag{48}$$

where terms which are finite as $x \to a$ have been neglected. In order to extract the singularities of the integral (48), the function $s_k(\xi)$ is expanded for small and large arguments. For small values of the argument

$$(b\xi)^{n+1/2}K_{n+1/2}(b\xi) \sim 2^{n-1/2}\Gamma(n+\tfrac{1}{2}), \tag{49}$$

and it follows that the integral (48) is finite at the lower limit. For large values of ξ

$$(b\xi)^{1/2}K_{n+1/2}(b\xi) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-b\xi}\left[1 + \frac{4(n+\frac{1}{2})^2}{8b\xi} + \cdots\right], \tag{50}$$

and eqn (43) implies

$$\xi s_k(\xi) \sim \frac{1}{1-\nu}\sum_{n=0}^{\infty}\left(\frac{\nu}{\nu-1}\right)^n = 1. \tag{51}$$

Accordingly, the dominant part of the integral in equation (48) yields

$$\tau(x) = -\tau_0\Phi(1)\left[1 - \frac{x}{(x^2 - a^2)^{1/2}}\right] + \cdots\cdots. \tag{52}$$

The expression for the shearing stress $\tau_{xz}(x, 0, 0)$ outside the crack can be obtained from eqns (9), (36), (38) and (52). Near the crack side the maximum shear stress may be expressed in the standard form

$$\tau_{xz}(x, 0, 0) = \frac{k_2}{(2\pi r)^{1/2}} + \cdots\cdots, \tag{53}$$

where r is a small distance measured from the side of the crack and the mode II stress-intensity factor is

$$k_2 = \Phi(1)\tau_0(\pi a)^{1/2}, \tag{53}$$

The numerical values of $k_2/\tau_0(\pi a)^{1/2}$ are shown in Table 1 for various aspect ratios of the crack sides and Poisson's ratios of the material of the solid. It is evident that there is a reduction in the value of k_2 for the narrow three-dimensional crack. For an infinite strip crack, $b \to \infty$, $\xi s_k(\xi) \to 1$, the kernel in eqn (30) becomes zero and the plane-strain solution is recovered.

Table 1. Values of $k_2/\tau_0(\pi a)^{1/2}$

$\frac{b}{a}$	ν			
	0.1	0.2	0.3	0.4
1	0.759	0.796	0.831	0.866
2	0.908	0.926	0.940	0.966
4	0.973	0.984	0.987	0.996
10	0.997	0.998	0.999	0.999

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